("Fundamenta Nova," p. 184), so that

$$\int_0^\infty xe^{-x^2}\mathrm{H}\left\{2x\left(\frac{n\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\left\{\tfrac{1}{2}\frac{n\mathrm{K}'}{\pi\mathrm{K}}k_{n+1}k'_{n+1}\mathrm{K}^3{}_{n+1}\right\}^{\frac{1}{2}} \quad . \quad (76).$$

Put $n=1$ and this formula gives

$$\int_0^\infty xe^{-x^2}\mathrm{H}\left\{2x\left(\frac{\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\tfrac{1}{2}\mathrm{K}k\left(\tfrac{1}{2}\frac{\mathrm{K}'k'^{\frac{1}{2}}}{\pi}\right)^{\frac{1}{2}} \quad . \quad . \quad (77).$$

IX. "Values of the Theta and Zeta Functions for certain Values of the Argument." By J. W. L. GLAISHER, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received July 31, 1879.

§ 1. In § 12 of the preceding paper "On Definite Integrals involving Elliptic Functions," it was necessary to determine the values of $\Theta(\frac{1}{2}\mathrm{K})$, $\Theta(\frac{1}{2}\mathrm{K}+i\mathrm{K}')$, $\Theta(\frac{1}{2}i\mathrm{K}')$, $\Theta(\mathrm{K}+\frac{1}{2}i\mathrm{K}')$, and $\Theta^4(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}')$, which were required in the evaluation of some of the integrals. This led me to calculate a table of the values of the Θ and H functions when the arguments were of the form $\mathrm{K}+ni\mathrm{K}'$, for the values 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, of m and n, and the results are contained in this paper. For the sake of completeness the corresponding values of the Z function are also given; and some remarks connected with the q-series to which the formulæ lead are added.

A table of the values of the sn, cn, dn for the above-mentioned arguments is given by Professor Cayley on page 74 of his "Elementary Treatise on Elliptic Functions" (1876): this table* is so useful that it seemed desirable to supplement it by a similar one for the Θ, H, and Z functions.

§ 2. The values found in § 13 are

$$\Theta(\tfrac{1}{2}\mathrm{K}) \qquad =\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}},$$

$$\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}') \quad =q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}(1+i),$$

$$\Theta(\tfrac{1}{2}i\mathrm{K}') \qquad =q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}},$$

* I may here note that the value of cn $(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}')$ should be $\frac{1-i}{\sqrt{2}}\frac{\sqrt{k'}}{\sqrt{k}}$ instead of $-\frac{1-i}{\sqrt{2}}\frac{\sqrt{k'}}{\sqrt{k}}$.

$$\Theta(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') \quad = q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}},$$

$$\Theta^4(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') = q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'+ik).$$

From this last equation the value of $\Theta(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}')$ may be deduced; but it is more convenient to avoid ambiguities of sign as follows.

In the formula

$$\Theta(u+v)\Theta(u-v) = \frac{\Theta^2 u\,\Theta^2 v}{\Theta^2 0}(1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v),$$

put $u=\frac{1}{2}\mathrm{K}$, $v=\frac{1}{2}i\mathrm{K}'$, then

$$\begin{aligned}\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\Theta(\tfrac{1}{2}\mathrm{K}-\tfrac{1}{2}i\mathrm{K}')&\\ &= \frac{\pi}{2k'\mathrm{K}}q^{-\frac{1}{8}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{4}}k'^{\frac{1}{4}}(1+k')^{\frac{1}{2}}(1-k)^{\frac{1}{2}}\left\{1-k^2\frac{1}{1+k'}\frac{i^2}{k}\right\}\\ &= q^{-\frac{1}{8}}\frac{\mathrm{K}}{k'\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\left(\frac{1-k}{1+k'}\right)^{\frac{1}{2}}(1+k+k')\\ &= q^{-\frac{1}{8}}\frac{2^{\frac{1}{2}}\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),\end{aligned}$$

since $\qquad 1+k+k'=\{2(1+k)(1+k')\}^{\frac{1}{2}}.$

Also, in the formula

$$\mathrm{dn}\,u = k'^{\frac{1}{2}}\frac{\Theta(u+\mathrm{K})}{\Theta u},$$

put $u=-\frac{1}{2}\mathrm{K}-\frac{1}{2}i\mathrm{K}'$, and

$$\Theta(\tfrac{1}{2}\mathrm{K}-\tfrac{1}{2}i\mathrm{K}') = \frac{(1+k')^{\frac{1}{2}}-i(1-k')^{\frac{1}{2}}}{2^{\frac{1}{2}}}\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') \quad . \quad . \quad (2),$$

whence from (1) and (2)

$$\Theta^2(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') = q^{-\frac{1}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1+k')^{\frac{1}{2}}+i(1-k')^{\frac{1}{2}}\}.$$

Extracting the square root, we have

$$\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') = q^{-\frac{1}{16}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}[\{2^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}+i\{2^{\frac{1}{2}}-(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}];$$

it is evident from the q-series for Θ (see § 8 of the present paper), that the sign of the real or imaginary parts is positive as above.

Since $\qquad \Theta^4(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}') = q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'+ik)$

$$= q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(\cos\theta+i\sin\theta),$$

where θ is the modular angle, it follows that

$$\Theta^2(\tfrac{1}{2}K+\tfrac{1}{2}iK')=q^{-\frac{1}{8}}\,\frac{2^{\frac{1}{2}}K}{\pi}\,k^{\frac{1}{4}}k'^{\frac{1}{4}}(\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta),$$

and
$$\Theta(\tfrac{1}{2}K+\tfrac{1}{2}iK')=q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}(\cos\tfrac{1}{4}\theta+i\sin\tfrac{1}{4}\theta).$$

This last result is readily identified with that found above, since

$$\cos\tfrac{1}{4}\theta=\frac{\{2^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},\qquad \sin\tfrac{1}{4}\theta=\frac{\{2^{\frac{1}{2}}-(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}}.$$

§ 3. From the values of Θu for $u=\frac{1}{2}K$, $\frac{1}{2}iK'$, $\frac{1}{2}K+\frac{1}{2}iK'$, the values of Θu for the other arguments contained in the table were deduced by means of the formulæ

$$\Theta(u+K)\ =k'^{-\frac{1}{2}}\operatorname{dn}u\,\Theta u,$$

$$\Theta(u+iK')=iq^{-\frac{1}{4}}k^{\frac{1}{2}}e^{-\frac{i\pi u}{2K}}\operatorname{sn}u\,\Theta u.$$

The values of Hu were deduced from those of Θu by the formula

$$Hu=-iq^{\frac{1}{4}}e^{\frac{i\pi u}{2K}}\Theta(u+iK').$$

For example,

$$\begin{aligned}H(\tfrac{1}{2}K+\tfrac{1}{2}iK')&=-iq^{\frac{1}{4}}e^{\frac{i\pi}{2K}(\frac{1}{2}K+\frac{1}{2}iK')}\,\Theta(\tfrac{1}{2}K+\tfrac{3}{2}iK')\\&=-iq^{\frac{1}{4}}e^{\frac{1}{4}i\pi}q^{\frac{1}{4}}\,.\,q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}(\sin\tfrac{1}{4}\theta+i\cos\tfrac{1}{4}\theta)\\&=q^{-\frac{1}{16}}\frac{K^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}(1-i)\,(\sin\tfrac{1}{4}\theta+i\cos\tfrac{1}{4}\theta)\\&=q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\cos\tfrac{1}{4}(\pi-\theta)+i\sin\tfrac{1}{4}(\pi-\theta)\},\end{aligned}$$

and it can be shown that

$$\cos\tfrac{1}{4}(\pi-\theta)=\frac{\{2^{\frac{1}{2}}+(1-k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},\qquad \sin\tfrac{1}{4}(\pi-\theta)=\frac{\{2^{\frac{1}{2}}-(1-k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}}.$$

The values in the table were verified throughout by the formulæ

$$\frac{Hu}{\Theta u}=k^{\frac{1}{2}}\operatorname{sn}u,\qquad \frac{H(u+K)}{\Theta u}=\frac{k^{\frac{1}{2}}}{k'^{\frac{1}{2}}}\operatorname{cn}u;$$

several of the results are of course deducible at once from one another by the relations $\Theta(u+2K)=\Theta u$, $H(u+2K)=-Hu$, and a change in the sign of i.

Values of Θu and $\mathrm{H}u$.

u.	Θu.	$\mathrm{H}u$.
$\frac{1}{2}\mathrm{K}$	$\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}}$	$\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}$
$\frac{3}{2}\mathrm{K}$	$\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}}$	$\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}$
$\frac{1}{2}\mathrm{K}+i\mathrm{K}'$	$q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}(1+i)$	$q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}}(1+i)$
$\frac{3}{2}\mathrm{K}+i\mathrm{K}'$	$q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}(1-i)$	$q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}}(1-i)$
$\frac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}}$	$iq^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}}$
$\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}}$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}}$
$\frac{3}{2}i\mathrm{K}'$	$-q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}}$	$iq^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}}$
$\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}}$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}}$
$\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\cos\frac{1}{4}\theta+i\sin\frac{1}{4}\theta\}$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\cos\frac{1}{4}(\pi-\theta)+i\sin\frac{1}{4}(\pi-\theta)\}$
$\frac{3}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\cos\frac{1}{4}\theta-i\sin\frac{1}{4}\theta\}$	$q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\cos\frac{1}{4}(\pi-\theta)-i\sin\frac{1}{4}(\pi-\theta)\}$
$\frac{1}{2}\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\sin\frac{1}{4}\theta+i\cos\frac{1}{4}\theta\}$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\sin\frac{1}{4}(\pi-\theta)+i\cos\frac{1}{4}(\pi-\theta)\}$
$\frac{3}{2}\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\sin\frac{1}{4}\theta-i\cos\frac{1}{4}\theta\}$	$q^{-\frac{9}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\{\sin\frac{1}{4}(\pi-\theta)-i\cos\frac{1}{4}(\pi-\theta)\}$

where θ is the modular angle, and

$$\cos\tfrac{1}{4}\theta=\frac{\{2^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},\qquad \sin\tfrac{1}{4}\theta=\frac{\{2^{\frac{1}{2}}-(1+k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},$$

$$\cos\tfrac{1}{4}(\pi-\theta)=\frac{\{2^{\frac{1}{2}}+(1-k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},\qquad \sin\tfrac{1}{4}(\pi-\theta)=\frac{\{2^{\frac{1}{2}}-(1-k')^{\frac{1}{2}}\}^{\frac{1}{2}}}{2^{\frac{3}{4}}},$$

§ 4. Also, squaring the values in the last group,

$$\Theta^2(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')=q^{-\frac{1}{8}}\frac{2^{\frac{1}{2}}\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta\},\ \text{\&c.},$$

so that we have the following table:—

u.	$\Theta^2 u$.	$\mathrm{H}^2 u$.
$\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1+k')^{\frac{1}{2}}+i(1-k')^{\frac{1}{2}}\}$	$q^{-\frac{1}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1-k')^{\frac{1}{2}}+i(1+k')^{\frac{1}{2}}\}$
$\tfrac{3}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1+k')^{\frac{1}{2}}-i(1-k')^{\frac{1}{2}}\}$	$q^{-\frac{1}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1-k')^{\frac{1}{2}}-i(1+k')^{\frac{1}{2}}\}$
$\tfrac{1}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}'$	$-q^{-\frac{9}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1+k')^{\frac{1}{2}}-i(1-k')^{\frac{1}{2}}\}$	$-q^{-\frac{9}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1-k')^{\frac{1}{2}}-i(1+k')^{\frac{1}{2}}\}$
$\tfrac{3}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}'$	$-q^{-\frac{9}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1+k')^{\frac{1}{2}}+i(1-k')^{\frac{1}{2}}\}$	$-q^{-\frac{9}{8}}\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}}\{(1-k')^{\frac{1}{2}}+i(1+k')^{\frac{1}{2}}\}$

and, squaring again,

u.	$\Theta^4 u$.	$\mathrm{H}^4 u$.
$\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'+ik)$	$q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(-k'+ik)$
$\tfrac{3}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$	$q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'-ik)$	$q^{-\frac{1}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(-k'-ik)$
$\tfrac{1}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}'$	$q^{-\frac{9}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'-ik)$	$q^{-\frac{9}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(-k'-ik)$
$\tfrac{3}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}'$	$q^{-\frac{9}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'+ik)$	$q^{-\frac{9}{4}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(-k'+ik)$

§ 5. In the case of the Z function, we have

$$\mathrm{Z}(\tfrac{1}{2}\mathrm{K})=\tfrac{1}{2}(1-k'),$$

$$\mathrm{Z}(\tfrac{1}{2}i\mathrm{K}')=-\tfrac{1}{4}\frac{i\pi}{\mathrm{K}}+\tfrac{1}{2}i(1+k),$$

$$\mathrm{Z}(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')=-\tfrac{1}{4}\frac{i\pi}{\mathrm{K}}+\tfrac{1}{2}(k+ik'),$$

and the following formulæ

$$\begin{aligned} Z(u+v) &= Zu+Zv-k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u+v), \\ Z(u+i\mathrm{K}') &= -\tfrac{1}{2}\frac{i\pi}{\mathrm{K}}+Zu+\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u}, \\ Z(u+2i\mathrm{K}') &= -\frac{i\pi}{\mathrm{K}}+Zu. \end{aligned}$$

From these we obtain the following table:—

Values of Zu.

u.	Zu.
$\frac{1}{2}\mathrm{K}$	$\frac{1}{2}(1-k')$
$\frac{3}{2}\mathrm{K}$	$-\frac{1}{2}(1-k')$
$\frac{1}{2}\mathrm{K}+i\mathrm{K}'$	$-\frac{1}{2}\frac{i\pi}{\mathrm{K}}+\frac{1}{2}(1+k')$
$\frac{3}{2}\mathrm{K}+i\mathrm{K}'$	$-\frac{1}{2}\frac{i\pi}{\mathrm{K}}-\frac{1}{2}(1+k')$
$\frac{1}{2}i\mathrm{K}'$	$-\frac{1}{4}\frac{i\pi}{\mathrm{K}}+\frac{1}{2}i(1+k)$
$\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$-\frac{1}{4}\frac{i\pi}{\mathrm{K}}+\frac{1}{2}i(1-k)$
$\frac{3}{2}i\mathrm{K}'$	$-\frac{3}{4}\frac{i\pi}{\mathrm{K}}-\frac{1}{2}i(1+k)$
$\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$-\frac{3}{4}\frac{i\pi}{\mathrm{K}}-\frac{1}{2}i(1-k)$
$\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$-\frac{1}{4}\frac{i\pi}{\mathrm{K}}+\frac{1}{2}(k+ik')$
$\frac{3}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$	$-\frac{1}{4}\frac{i\pi}{\mathrm{K}}-\frac{1}{2}(k-ik')$
$\frac{1}{2}\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$-\frac{3}{4}\frac{i\pi}{\mathrm{K}}+\frac{1}{2}(k-ik')$
$\frac{3}{2}\mathrm{K}+\frac{3}{2}i\mathrm{K}'$	$-\frac{3}{4}\frac{i\pi}{\mathrm{K}}-\frac{1}{2}(k+ik')$

§ 6. It is convenient also for the sake of completeness to place here the values of the functions for $u=0$, K, iK′, &c.

u.	Θu.	Hu.	Zu.
0	$\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	0	0
K	$\frac{2^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	$\frac{2^{\frac{1}{2}}k^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	0
iK'	0	$iq^{-\frac{1}{4}}\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	∞
$K+iK'$	$q^{-\frac{1}{4}}\frac{2^{\frac{1}{2}}k^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	$q^{-\frac{1}{4}}\frac{2^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	$-\frac{1}{2}\frac{i\pi}{K}$
$2K$	$\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	0	0
$2iK'$	$-q^{-1}\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	0	$-\frac{i\pi}{K}$[*]
$2K+iK'$	0	$-iq^{-\frac{1}{4}}\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	∞
$K+2iK'$	$q^{-1}\frac{2^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	$q^{-1}\frac{2^{\frac{1}{2}}k^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	$-\frac{i\pi}{K}$
$2K+2iK'$	$-q^{-1}\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$	0	$-\frac{i\pi}{K}$

§ 7. It follows from the values of Θu and Hu in § 3 that if

$$A'=\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}},$$

$$B'=\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}},$$

$$A=\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}},$$

$$B=\frac{2^{\frac{1}{4}}K^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}},$$

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[*] The value of Z($2i$K′) is by mistake given as 0 on page 164 of the "Fundamenta Nova."

then

$$\begin{array}{lll|lll}
\Theta(\tfrac{1}{2}\mathrm{K})=\Theta(\tfrac{3}{2}\mathrm{K}) & = & \mathrm{A}' & \mathrm{H}(\tfrac{1}{2}\mathrm{K})=(\mathrm{H}\tfrac{3}{2}\mathrm{K}) & = & \mathrm{B}' \\
\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}') & = & q^{-\frac{1}{4}}.\mathrm{B}'.\tfrac{1}{2}(1+i) & \mathrm{H}(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}') & = & q^{-\frac{1}{4}}.\mathrm{A}'.\tfrac{1}{2}(1+i) \\
\Theta(\tfrac{3}{2}\mathrm{K}+i\mathrm{K}') & = & q^{-\frac{1}{4}}.\mathrm{B}'.\tfrac{1}{2}(1-i) & \mathrm{H}(\tfrac{3}{2}\mathrm{K}+i\mathrm{K}') & = & q^{-\frac{1}{4}}.\mathrm{A}'.\tfrac{1}{2}(1-i) \\
 & & & & & \\
\Theta(\tfrac{1}{2}i\mathrm{K}') & = & q^{-\frac{1}{16}}.\mathrm{B} & \mathrm{H}(\tfrac{1}{2}i\mathrm{K}') & = & iq^{-\frac{1}{16}}.\mathrm{B} \\
\Theta(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') & = & q^{-\frac{1}{16}}.\mathrm{A} & \mathrm{H}(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') & = & q^{-\frac{1}{16}}.\mathrm{A} \\
\Theta(\tfrac{3}{2}i\mathrm{K}') & = & -q^{-\frac{9}{16}}.\mathrm{B} & \mathrm{H}(\tfrac{3}{2}i\mathrm{K}') & = & iq^{-\frac{9}{16}}.\mathrm{B} \\
\Theta(\mathrm{K}+\tfrac{3}{2}i\mathrm{K}') & = & q^{-\frac{9}{16}}.\mathrm{A} & \mathrm{H}(\mathrm{K}+\tfrac{3}{2}i\mathrm{K}') & = & q^{-\frac{9}{16}}.\mathrm{A}
\end{array}$$

Thus

$$\begin{aligned}
\mathrm{A}'&=\Theta(\tfrac{1}{2}\mathrm{K})=\Theta(\tfrac{3}{2}\mathrm{K}) \\
&=q^{\frac{1}{4}}(1-i)\mathrm{H}(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}')= && q^{\frac{1}{4}}(1+i)\mathrm{H}(\tfrac{3}{2}\mathrm{K}+i\mathrm{K}') \\
\mathrm{B}'&=\mathrm{H}(\tfrac{1}{2}\mathrm{K})=\mathrm{H}(\tfrac{3}{2}\mathrm{K}) \\
&=q^{\frac{1}{4}}(1-i)\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}')= && q^{\frac{1}{4}}(1+i)\Theta(\tfrac{3}{2}\mathrm{K}+i\mathrm{K}') \\
\mathrm{A}&=q^{\frac{1}{16}}\Theta(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') = && q^{\frac{9}{16}}\Theta(\mathrm{K}+\tfrac{3}{2}i\mathrm{K}') \\
&=q^{\frac{1}{16}}\mathrm{H}(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}') = && q^{\frac{9}{16}}\mathrm{H}(\mathrm{K}+\tfrac{3}{2}i\mathrm{K}') \\
\mathrm{B}&=q^{\frac{1}{16}}\Theta(\tfrac{1}{2}i\mathrm{K}') = && -q^{\frac{9}{16}}\Theta(\tfrac{3}{2}i\mathrm{K}') \\
&=-iq^{\frac{1}{16}}\mathrm{H}(\tfrac{1}{2}i\mathrm{K}') = && -iq^{\frac{9}{16}}\mathrm{H}(\tfrac{3}{2}i\mathrm{K}')
\end{aligned}$$

whence

$$\frac{\mathrm{B}'}{\mathrm{A}'}=\frac{\mathrm{H}(\frac{1}{2}\mathrm{K})}{\Theta(\frac{1}{2}\mathrm{K})}=\frac{\mathrm{H}(\frac{3}{2}\mathrm{K})}{\Theta(\frac{3}{2}\mathrm{K})}=\frac{\Theta(\frac{1}{2}\mathrm{K}+i\mathrm{K}')}{\mathrm{H}(\frac{1}{2}\mathrm{K}+i\mathrm{K}')}=\frac{\Theta(\frac{3}{2}\mathrm{K}+i\mathrm{K}')}{\mathrm{H}(\frac{3}{2}\mathrm{K}+i\mathrm{K}')}$$

$$\frac{\mathrm{B}}{\mathrm{A}}=\frac{\Theta(\frac{1}{2}i\mathrm{K}')}{\Theta(\mathrm{K}+\frac{1}{2}i\mathrm{K}')}=-\frac{\Theta(\frac{3}{2}i\mathrm{K}')}{\Theta(\mathrm{K}+\frac{3}{2}i\mathrm{K}')}=-i\frac{\mathrm{H}(\frac{1}{2}i\mathrm{K}')}{\mathrm{H}(\mathrm{K}+\frac{1}{2}i\mathrm{K}')}=-i\frac{\mathrm{H}(\frac{3}{2}i\mathrm{K}')}{\mathrm{H}(\mathrm{K}+\frac{3}{2}i\mathrm{K}')},$$

equations which admit of immediate verification.

Also, if $$\mathrm{C}=\frac{\mathrm{K}}{\pi}k^{\frac{1}{4}}k'^{\frac{1}{4}},$$

then
$$\begin{aligned}
\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\mathrm{H}(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')&= q^{-\frac{1}{8}}\mathrm{C}(1+i), \\
\Theta(\tfrac{3}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\mathrm{H}(\tfrac{3}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')&= q^{-\frac{1}{8}}\mathrm{C}(1-i), \\
\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}')\mathrm{H}(\tfrac{1}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}')&=-q^{-\frac{1}{8}}\mathrm{C}(1-i), \\
\Theta(\tfrac{3}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}')\mathrm{H}(\tfrac{3}{2}\mathrm{K}+\tfrac{3}{2}i\mathrm{K}')&=-q^{-\frac{1}{8}}\mathrm{C}(1+i).
\end{aligned}$$

§ 8. The q-series for Θ and H are

$$\Theta u=1-2q\cos\frac{\pi u}{\mathrm{K}}+2q^4\cos\frac{2\pi u}{\mathrm{K}}-2q^9\cos\frac{3\pi u}{\mathrm{K}}+\ \&\mathrm{c.},$$

$$\mathrm{H}u=2q^{\frac{1}{4}}\sin\frac{\pi u}{2\mathrm{K}}-2q^{\frac{9}{4}}\sin\frac{3\pi u}{2\mathrm{K}}+2q^{\frac{25}{4}}\sin\frac{5\pi u}{\mathrm{K}}-\ \&\mathrm{c.},$$

and comparing the q-series for $\Theta(\frac{1}{2}\mathrm{K})$, &c., with their values A′, &c., given at the beginning of § 7, we have the following equations:—

$$1-2q^4+2q^{16}-2q^{36}+\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}}, \quad . \quad . \quad . \quad (3),$$

$$q^{\frac{1}{4}}-q^{\frac{9}{4}}-q^{\frac{25}{4}}+q^{\frac{49}{4}}+q^{\frac{81}{4}}-\text{ \&c. }=\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}. \quad . \quad . \quad . \quad (4),$$

$$q^{\frac{1}{16}}+q^{\frac{9}{16}}+q^{\frac{25}{16}}+q^{\frac{49}{16}}+q^{\frac{81}{16}}+\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}} \quad . \quad . \quad . \quad (5),$$

$$q^{\frac{1}{16}}-q^{\frac{9}{16}}-q^{\frac{25}{16}}+q^{\frac{49}{16}}+q^{\frac{81}{16}}-\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}} \quad . \quad . \quad . \quad (6),$$

the signs of the terms in (4) and (6) being, after the first, alternately negative and positive in pairs.

Of these (3) is deducible at once from

$$1-2q+2q^4-2q^9+\text{ \&c. }=\Theta 0=\frac{2^{\frac{1}{2}}k'^{\frac{1}{2}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$$

by the transformation of q into q^2 twice repeated. The same double transformation converts (6) into (4), and (5) into the formula

$$q^{\frac{1}{4}}+q^{\frac{9}{4}}+q^{\frac{25}{4}}+q^{\frac{49}{4}}+\text{ \&c. }=\frac{k^{\frac{1}{2}}\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}};$$

so that (3), . . . (6) may be readily obtained independently of the values of $\Theta(\frac{1}{2}\mathrm{K})$, &c.

If the modulus be changed from k to k', (6) becomes

$$r^{\frac{1}{16}}-r^{\frac{9}{16}}-r^{\frac{25}{16}}+r^{\frac{49}{16}}+r^{\frac{81}{16}}-\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}'^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}},$$

where $r=e^{-\frac{\pi\mathrm{K}}{\mathrm{K}'}}$.

Thus
$$\frac{q^{\frac{1}{4}}-q^{\frac{9}{4}}-q^{\frac{25}{4}}+q^{\frac{49}{4}}+q^{\frac{81}{4}}-\text{ \&c.}}{r^{\frac{1}{16}}-r^{\frac{9}{16}}-r^{\frac{25}{16}}+r^{\frac{49}{16}}+r^{\frac{81}{16}}-\text{ \&c.}}=\frac{1}{2^{\frac{1}{2}}}\left(\frac{\mathrm{K}}{\mathrm{K}'}\right)^{\frac{1}{2}},$$

so that if q, r be any quantities less than unity connected by the relation

$$\log\frac{1}{q}\log\frac{1}{r}=\pi^2,$$

then
$$\left(\log\frac{1}{q}\right)^{\frac{1}{4}}(q^{\frac{1}{4}}-q^{\frac{9}{4}}-q^{\frac{25}{4}}+q^{\frac{49}{4}}+q^{\frac{81}{4}}-\text{ \&c.})$$

$$=\frac{1}{2^{\frac{1}{2}}}\left(\log\frac{1}{r}\right)^{\frac{1}{4}}(r^{\frac{1}{16}}-r^{\frac{9}{16}}-r^{\frac{25}{16}}+r^{\frac{49}{16}}+r^{\frac{81}{16}}-\text{ \&c.}),$$

which may also be written

$$e^{-a^2}-e^{-9a^2}-e^{-25a^2}+e^{-49a^2}+\text{ \&c.}$$

$$=\frac{\pi^{\frac{1}{2}}}{2^{\frac{3}{2}}a}\left(e^{-\frac{1}{64}\frac{\pi^2}{a^2}}-e^{-\frac{9}{64}\frac{\pi^2}{a^2}}-e^{-\frac{25}{64}\frac{\pi^2}{a^2}}+e^{-\frac{49}{64}\frac{\pi^2}{a^2}}+\text{ \&c.}\right),$$

or
$$\mu^{\frac{1}{4}}(e^{-\frac{1}{8}\mu}-e^{-\frac{9}{8}\mu}-e^{-\frac{25}{8}\mu}+e^{-\frac{49}{8}\mu}+\text{ \&c.})$$
$$=\nu^{\frac{1}{4}}(e^{-\frac{1}{8}\nu}-e^{-\frac{9}{8}\nu}-e^{-\frac{25}{8}\nu}+e^{-\frac{49}{8}\nu}+\text{ \&c.}) \quad . \quad . \quad . \quad . \quad (7),$$

where μ, ν are any quantities connected by the relation

$$\mu\nu=\pi^2.$$

Changing q into q^2, (6) becomes

$$q^{\frac{1}{8}}-q^{\frac{9}{8}}-q^{\frac{25}{8}}+q^{\frac{49}{8}}+\text{ \&c. }=\frac{\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{4}}k'^{\frac{1}{4}},$$

which also leads to (7). This identity is of the same kind as the well-known formula

$$\mu^{\frac{1}{4}}(1+2e^{-\mu}+2e^{-4\mu}+2e^{-9\mu}+\text{ \&c.})=\nu^{\frac{1}{4}}(1+2e^{-\nu}+2e^{-4\nu}+2e^{-9\nu}+\text{ \&c.}),$$

viz., in which a function of μ is equated to the same function of $\frac{\pi^2}{\mu}$. Identities of this class form the subject of a paper in the fifth volume of the "Messenger of Mathematics,"* pp. 174—179; and the identity which results from (3) and (5), viz.:

$$2\mu^{\frac{1}{4}}(e^{-\frac{1}{4}\mu}+e^{-\frac{9}{4}\mu}+e^{-\frac{25}{4}\mu}+\text{ \&c.})=\nu^{\frac{1}{4}}(1-2e^{-\nu}+2e^{-4\nu}-2e^{-9\nu}+\text{ \&c.})$$

is given on page 177 of that paper.

§ 9. From the third group of values in § 3 it might be shown that

$$q^{\frac{1}{16}}-q^{\frac{49}{16}}-q^{\frac{81}{16}}+q^{\frac{225}{16}}+q^{\frac{289}{16}}-\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\cos\tfrac{1}{4}\theta \quad . \quad . \quad (8),$$

$$q^{\frac{9}{16}}-q^{\frac{25}{16}}-q^{\frac{121}{16}}+q^{\frac{169}{16}}+q^{\frac{361}{16}}-\text{ \&c. }=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}k'^{\frac{1}{8}}\sin\tfrac{1}{4}\theta \quad . \quad . \quad (9),$$

where θ denotes the modular angle. We thus have

$$\tan\tfrac{1}{4}\theta=\frac{q^{\frac{9}{16}}-q^{\frac{25}{16}}-q^{\frac{121}{16}}+q^{\frac{169}{16}}+q^{\frac{361}{16}}-\text{ \&c.}}{q^{\frac{1}{16}}-q^{\frac{49}{16}}-q^{\frac{81}{16}}+q^{\frac{225}{16}}+q^{\frac{289}{16}}-\text{ \&c.}}.$$

It is a known theorem† that

$$\tfrac{1}{4}\theta=\text{arc tan } q^{\frac{1}{2}}-\text{arc tan } q^{\frac{3}{2}}+\text{arc tan } q^{\frac{5}{2}}-\text{ \&c.},$$

so that
$$\text{arc tan } q^{\frac{1}{2}}-\text{arc tan } q^{\frac{3}{2}}+\text{arc tan } q^{\frac{5}{2}}-\text{ \&c.}$$
$$=\text{arc tan}\frac{q^{\frac{9}{16}}-q^{\frac{25}{16}}-q^{\frac{121}{16}}+q^{\frac{169}{16}}+q^{\frac{361}{16}}-\text{ \&c.}}{q^{\frac{1}{16}}-q^{\frac{49}{16}}-q^{\frac{81}{16}}+q^{\frac{225}{16}}+q^{\frac{289}{16}}-\text{ \&c.}}.$$

In the "Philosophical Magazine"‡ for December, 1875, it is shown that

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* "Notes on certain Formulæ in Jacobi's 'Fundamenta Nova'" (March and April, 1876).

† "Fundamenta Nova," p. 108.

‡ "On some Identities derived from Elliptic-Function Formulæ," ser. 4, vol. 50, pp. 539—542.

$$\cos(\text{arc}\tan q - \text{arc}\tan q^3 + \text{arc}\tan q^5 - \text{\&c.})$$

$$= \frac{1 - q^6 - q^{10} + q^{28} + q^{36} - \text{\&c.}}{(1+q^2)^{\frac{1}{2}}(1+q^6)^{\frac{1}{2}} \ldots 1 - q^4 . 1 - q^8 \ldots} \quad . \quad . \quad . \quad (10),$$

$$\sin(\text{arc}\tan q - \text{arc}\tan q^3 + \text{arc}\tan q^5 - \text{\&c.})$$

$$= \frac{q - q^3 - q^{15} + q^{21} + q^{45} - \text{\&c.}}{(1+q^2)^{\frac{1}{2}}(1+q^6)^{\frac{1}{2}} \ldots 1 - q^4 . 1 - q^8 \ldots} \quad . \quad . \quad . \quad (11),$$

the exponents being the triangular numbers. These formulæ can be readily identified with (8) and (9); for

$$(1 + q . 1 + q^3 \ldots)^{\frac{1}{2}} = \frac{2^{\frac{1}{12}} q^{\frac{1}{48}}}{k^{\frac{1}{24}} k'^{\frac{1}{24}}},$$

$$(1 - q^2 . 1 - q^4 \ldots) = \frac{2^{\frac{1}{6}} \mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} k^{\frac{1}{6}} k'^{\frac{1}{6}} q^{-\frac{1}{12}},$$

so that, replacing q by $q^{\frac{1}{2}}$ and putting for

$$\text{arc}\tan q^{\frac{1}{2}} - \text{arc}\tan q^{\frac{3}{2}} + \text{arc}\tan q^{\frac{5}{2}} - \text{\&c.},$$

its value $\frac{1}{4}\theta$, (10) and (11) become

$$\cos\tfrac{1}{4}\theta = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{4}} \mathrm{K}^{\frac{1}{2}}} \frac{q^{\frac{1}{16}}}{k^{\frac{1}{8}} k'^{\frac{1}{8}}} (1 - q^3 - q^5 + q^{14} + q^{18} - \text{\&c.}),$$

$$\sin\tfrac{1}{4}\theta = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{4}} \mathrm{K}^{\frac{1}{2}}} \frac{q^{\frac{1}{16}}}{k^{\frac{1}{8}} k'^{\frac{1}{8}}} (q^{\frac{1}{2}} - q^{\frac{3}{2}} - q^{\frac{15}{2}} + q^{\frac{21}{2}} + q^{\frac{45}{2}} - \text{\&c.}),$$

which are in fact (8) and (9).

X. "On certain Definite Integrals." No. 5. By W. H. L. RUSSELL, F.R.S. Received October 13, 1879.

The following is a continuation of four papers inserted in the "Proceedings of the Royal Society." In my last paper I gave a proof that

$$\sin(2n+3)\theta - \sin(2n+1)\theta = 0$$

when (n) is infinite. We may consider the subject also thus. Taking the expression

$$\frac{(2n+3)\theta - \frac{(2n+3)^3}{1 . 2 . 3}\theta^3 + \ldots \pm \frac{(2n+3)^{2m+1}}{1 . 2 \ldots 2m+1}\theta^{2m+1}}{(2n+1)\theta - \frac{(2n+1)^3}{1 . 2 . 3}\theta^3 \quad \ldots \pm \frac{(2n+1)^{2m+1}}{1 . 2 . 3 . 2m+1}\theta^{2m+1}}$$

we see manifestly that as (n) increases to infinity the fraction becomes unity. And this is true however large (m) is considered to be.